Understanding the Euler Condition

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Let's consider the Euler equation in the following model.

Model:
The representative household (infinitely lived) solves a consumption-savings problem to maximise their expected lifetime utility,

$$\max_{\{c_t,k_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Their per-period utility function is,

$$u(c) = \begin{cases} \dfrac{c^{1-\sigma}-1}{1-\sigma}, & \sigma \neq 1, \\ \ln c, & \sigma = 1, \end{cases} \qquad \sigma > 0.$$

Capital (savings) chosen at time t becomes productive (investment) in period $t+1$:

$$y_t = Z_t k_{t-1}^{\alpha}, \qquad 0 < \alpha < 1$$

where Z_{t+1} follows some stochastic process , e.g. AR(1). *(details not needed for the Euler equation)*

The resource constraint is as follow. Output plus undepreciated capital (from last period) can be used for consumption and to carry capital into next period:

$$c_t + k_t = y_t + (1-\delta)k_{t-1} = Z_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1}, \qquad 0 < \delta < 1.$$

Derivation:
Next, you can easily derive Euler equation by taking Lagrangian and FOCs

$$\mathscr{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[u(c_t) + \lambda_t \big(Z_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - c_t - k_t \big) \Big].$$

$$\frac{\partial \mathscr{L}}{\partial c_t}: \quad u'(c_t) - \lambda_t = 0 \;\Rightarrow\; \lambda_t = u'(c_t). \tag{FOC-c}$$

$$\frac{\partial \mathscr{L}}{\partial k_t}: \quad -\lambda_t + \beta\,\mathbb{E}_t\Big[\lambda_{t+1}\big(\alpha Z_{t+1}k_t^{\alpha-1} + (1-\delta)\big)\Big] = 0. \tag{FOC-k}$$

Substitute $\lambda_t = u'(c_t)$ and $\lambda_{t+1} = u'(c_{t+1})$ into (FOC-k):

$$u'(c_t) = \beta\,\mathbb{E}_t\Big[u'(c_{t+1})\big(\alpha Z_{t+1}k_t^{\alpha-1} + 1 - \delta\big)\Big]. \tag{Euler equation}$$

For CRRA, $u'(c) = c^{-\sigma}$. Divide both sides by $u'(c_t) = c_t^{-\sigma}$:

$$1 = \beta\,\mathbb{E}_t\Big[\frac{u'(c_{t+1})}{u'(c_t)}\big(\alpha Z_{t+1}k_t^{\alpha-1} + 1 - \delta\big)\Big] = \beta\,\mathbb{E}_t\Big[\Big(\frac{c_t}{c_{t+1}}\Big)^{\sigma}\big(\alpha Z_{t+1}k_t^{\alpha-1} + 1 - \delta\big)\Big],$$

1 The Euler Equation

The first equilibrium condition is the Euler equation:

$$1 = \beta\mathbb{E}_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\left(\alpha Z_{t+1}k_t^{\alpha-1} + 1 - \delta\right)\right]. \tag{1}$$

Notice that

$$\left(\frac{c_t}{c_{t+1}}\right)^{\sigma} = \frac{u'(c_{t+1})}{u'(c_t)},$$

since under CRRA utility,

$$u'(c_t) = c_t^{-\sigma}.$$

Rewriting the Euler equation by multiplying both sides by $u'(c_t)$:

$$u'(c_t) = \beta\mathbb{E}_t\left[\left(\alpha Z_{t+1}k_t^{\alpha-1} + 1 - \delta\right)u'(c_{t+1})\right]. \tag{2}$$

This condition equates the marginal utility cost of giving up one unit of consumption today to the discounted expected marginal utility gain from investing that unit and consuming the return tomorrow.

1.1 Benchmark Case: $\beta = 1$ and No Capital Return

Suppose first that $\beta = 1$ and ignore the capital return term. Then the Euler equation simplifies to:

$$u'(c_t) = u'(c_{t+1}).$$

Since utility is strictly increasing and concave, this implies:

$$c_t = c_{t+1}.$$

Thus, without discounting or returns, the household smooths consumption perfectly across time.

1.2 Role of β (patience / time preference)

Now suppose $0 < \beta < 1$.
The Euler equation becomes (no capital return term):

$$u'(c_t) = \beta \mathbb{E}_t[u'(c_{t+1})],$$

Because $\beta < 1$, future marginal utility is discounted. To satisfy the equality, we must have:

$$u'(c_{t+1}) > u'(c_t),$$

which implies:

$$c_{t+1} < c_t.$$

This reflects time preference: the household values current consumption more than future consumption.

1.3 Economic Interpretation of the Euler Equation

The term

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

is the **stochastic discount factor (SDF)**. It measures the discounted marginal value of future consumption in units of current marginal utility.

The return term

$$R_{t+1} = \alpha Z_{t+1} k_t^{\alpha-1} + 1 - \delta$$

represents the marginal product of capital plus undepreciated capital.

The Euler equation therefore says:

$$1 = \mathbb{E}_t[m_{t+1} R_{t+1}].$$

If the household reduces consumption by one unit today and invests it, tomorrow it obtains R_{t+1} units of goods. The optimality condition requires that the expected discounted marginal benefit equals the marginal cost.

1.4 Graphical Intuition: Capital Policy Function

Figure 1 illustrates the capital policy function $k_{t+1} = g(k_t)$. The blue solid curve shows the optimal policy function implied by the Euler equation. The 45-degree line represents points where $k_{t+1} = k_t$, i.e. where capital remains constant over time.

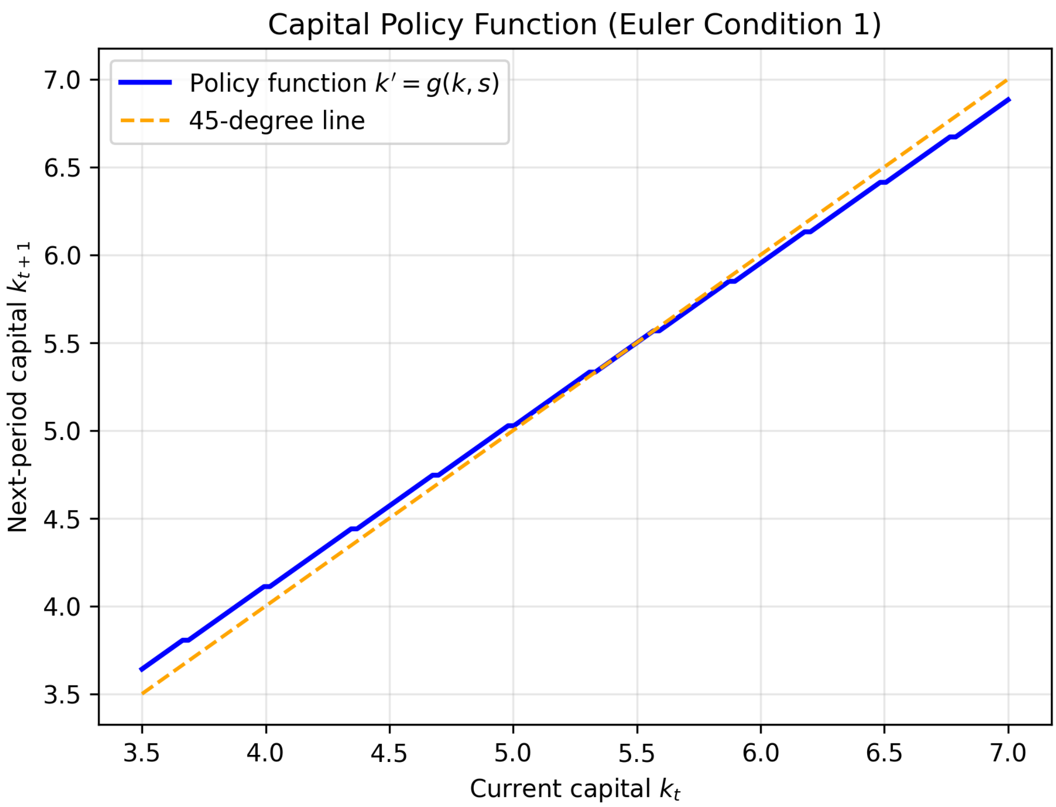

Figure 1: Capital policy function under CRRA utility

- When k_t is low, the marginal product of capital (MPK) is high. Since the return to investment is large, households choose to save more, implying $k_{t+1} > k_t$. Capital therefore increases over time.
- When k_t is high, MPK is low. Investment becomes less attractive, so households reduce savings, implying $k_{t+1} < k_t$. Capital decreases over time.
- The intersection between the policy function and the 45-degree line determines the steady state.

At this stage, we clearly understand the role of MPK in shaping the slope of the policy function. However, an important question arises:

If MPK explains why investment changes, what is the role of the stochastic discount factor m_{t+1}?

Recall the Euler equation:

$$1 = \mathbb{E}_t \left[m_{t+1} R_{t+1} \right],$$

The term R_{t+1} captures the physical return to capital. It reflects technological productivity and depreciation.

The term m_{t+1}, instead, captures the intertemporal valuation of consumption. It arises from the concavity of utility and measures how much the household values future consumption relative to current consumption.

To answer this question better, let's consider a counterfactual case.

1.5 Counterfactual: Linear Utility

Now consider a counterfactual case where utility is linear:

$$u(c) = c.$$

Then:

$$u'(c) = 1,$$

so the stochastic discount factor becomes:

$$m_{t+1} = \beta.$$

The Euler equation simplifies to:

$$1 = \beta \mathbb{E}_t[R_{t+1}].$$

In this case, intertemporal trade-offs depend only on the return to capital. There is no diminishing marginal utility to generate consumption smoothing.

Graphically, Figure 2 demonstrates that the policy function becomes horizontal beyond the point where $\beta R_{t+1} = 1$.

When capital is small, MPK is high, so households invest more. But once the optimal MPK is reached, additional income is entirely consumed rather than invested further.

1.6 Why the Stochastic Discount Factor Matters

Now, it is clear why the SDF matters. The term m_{t+1} arises from the concavity of utility.

Because utility is concave:

$$u''(c) < 0,$$

households dislike large fluctuations in consumption. They prefer to smooth consumption over time.

Even if investing increases future consumption, saving reduces current consumption. If current consumption becomes too low relative to future consumption, the marginal utility loss today outweighs the gain tomorrow.

Remember that total resources must satisfy:

$$c_t + k_{t+1} = Z_t k_t^{\alpha} + (1-\delta)k_t + d_t.$$

Higher savings today necessarily implies lower current consumption.

Therefore, the Euler equation captures two opposing forces:

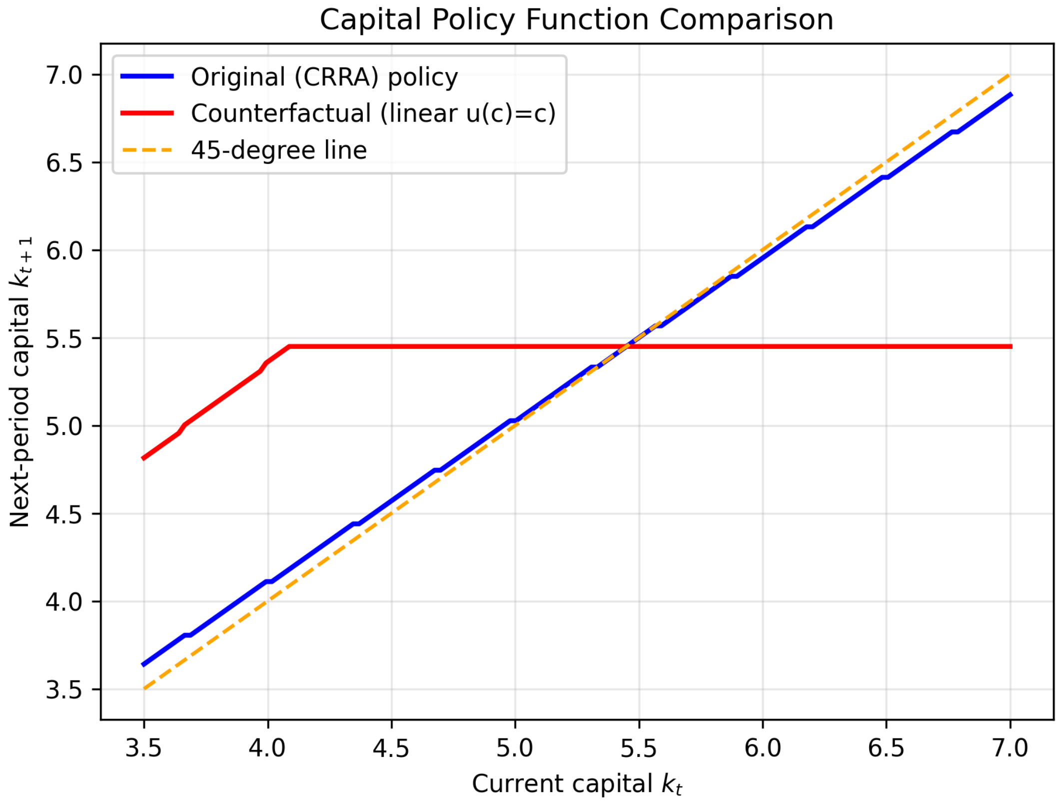

Figure 2: Capital policy function under linear utility

- The marginal product of capital (MPK), which encourages saving.
- The marginal utility of consumption, which encourages smoothing.

Without the stochastic discount factor (linear utility case), households would invest up to the point where returns equal the discount factor, and consume all additional resources. Because marginal utility is constant in this case. There is no consumption smoothing motive, as giving up one unit of consumption today always entails the same utility cost, regardless of the consumption level. With concave utility, households balance returns against intertemporal consumption smoothing.